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UNIQUENESS OF INITIAL-BOUNDARY VALUE PROBLEMS IN NONLOCAL ELASTICITY

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Abstract—In this study we have dealt with the uniqueness of the solutions of a class of initial-boundary value problems in linear, isotropic, homogeneous, nonlocal elasticity. The proof of uniqueness is based on the positive definiteness of total strain energy. After giving the sufficient conditions under which total strain energy is positive definite, it has been shown that the solutions of the initial-boundary value problems which are considered in this study are unique. Existence of the solutions of these problems is an *a priori* assumption.

1. INTRODUCTION

One of the main streams of the advancement of science is to expand the extent of the fundamental hypotheses of a theory when the theory proves insufficient in explaining the problems in its field. Although these enlargements bring a lot of complications the efforts go on for the sake of being able to explain more phenomena. As well as in the other branches of science, in continuum mechanics there are many researches made in this direction. Among them, the nonlocal theory of continuum mechanics is perhaps the most recent one.

As well as the other nonlocal theories in continuum mechanics, the nonlocal theory of elasticity is also of recent origin and differs from the local one in fundamental hypotheses. As is well known, in the classical theory of elasticity, the balance laws are postulated to be valid in any portion to be cut from the body. In nonlocal elasticity, this postulate is abandoned and the balance laws are assumed to be valid only on the whole of the body. As a result of this approach the constitutive equations of nonlocal elasticity appear as integral equations, in terms of strain tensor, either the *Fredholm equation of first kind*

$$t_{ij}(\mathbf{x},t) = \int_B \alpha(|\mathbf{x}-\mathbf{x}'|)\{\lambda\varepsilon_{kk}(\mathbf{x}',t)\delta_{ij}+2\mu\varepsilon_{ij}(\mathbf{x}',t)\}\,\mathrm{d}v(\mathbf{x}') \tag{1}$$

or the *Fredholm equation of second kind*

$$t_{ij}(\mathbf{x},t) = \lambda\varepsilon_{kk}(\mathbf{x},t)\delta_{ij}+2\mu\varepsilon_{ij}(\mathbf{x},t)+\gamma(\xi)\int_B \beta(|\mathbf{x}-\mathbf{x}'|,\xi)\{\lambda\varepsilon_{kk}(\mathbf{x}',t)\delta_{ij}+2\mu\varepsilon_{ij}(\mathbf{x}',t)\}\,\mathrm{d}v(\mathbf{x}') \tag{2}$$

for linear, isotropic, homogeneous materials. For the physical background of the constitutive equations of nonlocal elasticity Kroner (1967), Kunin (1983) and Rogula (1982) can be consulted. Similar results are obtained by Eringen and Edelen (1972) by employing thermomechanical and variational approaches. To describe the kernels (they are called an *interaction kernel* in nonlocal mechanics) appearing in these equations, the dispersion relations which are obtained by lattice dynamics, are employed (Eringen, 1972, 1973). The problems solved in the frame of nonlocal elasticity indicate the power of this approach (Eringen *et al.*, 1977; Eringen, 1978).

Since the nonlocal theory of elasticity is of recent origin it is open to study from many points of view. Perhaps, the most important question is whether the problems defined in

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the frame of nonlocal elasticity are well-posed. For this purpose, we find it very useful to study the conditions under which the solutions of the problems which are defined in nonlocal elasticity exist and are unique.

The uniqueness of the solutions of the boundary value problems in classical elasticity was first studied by Kirchhoff† in the 1850s. Kirchhoff has shown that if the strain energy is non-negative then a boundary value problem in elasticity possesses at most one solution, if it exists. Moreover Kirchhoff has also shown that if the elastic constants λ and μ obey the inequalities

$$3\lambda+2\mu > 0, \quad \mu > 0 \tag{3}$$

then the strain energy stored in the body due to a three-dimensional displacement field is non-negative. In the linear theory of elastodynamics the analogous result due to Neumann‡ is that the initial-boundary value problems possess only one solution provided that the elastic constants satisfy the inequalities (3). For an extensive account on the uniqueness theorems in linear elasticity Knops and Payne (1971) can be consulted.

The main goal of this study is to construct a uniqueness theorem for the initial-boundary value problem of homogeneous, isotropic, linear nonlocal elastodynamics. In a previous work of the author (Altan, 1984) an analogous theorem has been given where the constitutive relation was of the form (1). In this study the stress–strain relation is taken to be (2).

In the following section notation and some mathematical preliminaries are introduced. In Section 3 the initial-boundary value problems considered in this study are defined and the conditions under which the total strain energy is non-negative, are obtained. In Section 4 it is shown that the solutions of the considered initial-boundary value are unique. Finally, a short discussion on the constitutive relations (1) and (2) is given.

2. NOTATION AND SOME PRELIMINARIES

Throughout this paper Cartesian coordinates and conventional indicial notation are used. Subscripts will have the range of the integers 1, 2, 3 and denote the Cartesian components of a tensor-valued function. Repeated indices imply the summation over the range and the indices following a comma indicate partial differentiations with respect to coordinates. A superposed dot is used for time derivative. B is an open, bounded region in three-dimensional Euclidean space which is occupied by the body. The elements of this space are denoted by $\mathbf{x}$ (position vector). The boundary of the closure of B is shown by S and for the components of the unit outward normal of S the notation n_i is used. To avoid some geometrical complications the region B and the surface S are assumed to be sufficiently smooth so that the Green–Gauss identity is valid:

$$\int_B \operatorname{div} \mathbf{V}\, dv = \int_S \mathbf{V}\cdot\mathbf{n}\, da \tag{4}$$

where $\mathbf{V}$ is a continuous tensor-valued function. u_i, ε_{ij}, t_{ij} are used to denote the Cartesian components of infinitesimal displacement vector, strain and stress tensors, respectively. $\rho(\mathbf{x})$ and f_i are the body forces and mass density, $\rho(\mathbf{x}) > 0$ is assumed.

The following preliminaries are very useful for the purposes of this study: Let $K(\mathbf{x},\mathbf{z})$ be a real-valued, continuous, symmetric and square integrable function in $B\times B$, i.e.

$$K(\mathbf{x},\mathbf{z}) = K(\mathbf{z},\mathbf{x}) \quad \text{for all} \quad \mathbf{x},\mathbf{z}\in B \tag{5}$$

and

† G. R. Kirchhoff, German scientist and mathematician (1824–1887).
‡ F. E. Neumann, German scientist and mathematician (1798–1895).

$$\int_B \int_B |K(\mathbf{x},\mathbf{z})|^2 \, dv(\mathbf{x}) \, dv(\mathbf{z}) = C^2 < \infty. \tag{6}$$

Such a function can be expressed as an infinite (or finite, if the kernel is degenerate) sum almost everywhere in $B \times B$:

$$K(\mathbf{x},\mathbf{z}) = \sum_{j=0}^{\infty} \kappa_j^{-1} \Phi_j(\mathbf{x}) \Phi_j(\mathbf{z}). \tag{7}$$

Here $\{\Phi_j(\mathbf{x})\}_{j=0}^{\infty}$ is an orthonormal base which should not be necessarily complete in square integrable function space $\{L_2(B)\}$ and $\{\kappa_j\}_{j=0}^{\infty}$ is a monotone increasing sequence of real numbers. Let $\{\Psi_j(\mathbf{x})\}_{j=0}^{\infty}$ be a complete orthonormal base in $L_2(B)$ including all of the elements of the set $\{\Phi_j(\mathbf{x})\}_{j=0}^{\infty}$. As is well known, any arbitrary function $f(\mathbf{x}) \in L_2(B)$ can be expressed to be an infinite sum in terms of $\Psi_j(\mathbf{x})$,

$$f(\mathbf{x}) = \sum_{j=0}^{\infty} f_j \Psi_j(\mathbf{x}) \tag{8}$$

where

$$f_j = \int_B f(\mathbf{x}) \Psi_j(\mathbf{x}) \, dv. \tag{9}$$

For more information about the integral equation with symmetric kernels consult Pogorzelski (1966).

3. INITIAL-BOUNDARY VALUE PROBLEMS IN NONLOCAL ELASTICITY

In this section we wish to introduce a class of initial-boundary value problems in nonlocal elasticity. Since the field equations of nonlocal elasticity are different from the classical ones, definitions of the initial-boundary problems in nonlocal elasticity should be studied separately. Such studies will be very useful for the improvement of nonlocal theory. The main purpose of this study is to discuss the uniqueness of a class of initial-boundary boundary problems whose definitions are in accordance with the classical ones, rather than to discuss the definitions of the initial-boundary value problems in nonlocal elasticity. Moreover, the initial-boundary value problems defined in this study are compatible with the other studies in nonlocal elasticity: Iesan (1977), Eringen *et al.* (1977), Eringen and Balta (1979), Eringen (1979) and recently Eringen (1987). Ari (1982) indicated that the mixed boundary conditions in nonlocal elasticity should be treated in a different way but we will not consider them in this study.

The field equations of homogeneous, isotropic, linear, nonlocal elasticity consist of *the displacement–strain relations*

$$2\varepsilon_{ij}(\mathbf{x},t) = u_{i,j}(\mathbf{x},t) + u_{j,i}(\mathbf{x},t) \tag{10}$$

the equations of motions

$$t_{ij,i}(\mathbf{x},t) + f_j(\mathbf{x},t) = \rho(\mathbf{x}) \ddot{u}_j(\mathbf{x},t), \quad t_{ij} = t_{ji} \tag{11}$$

and *the stress–strain relations*

$$t_{ij}(\mathbf{x},t) = \lambda \varepsilon_{kk}(\mathbf{x},t)\delta_{ij} + 2\mu\varepsilon_{ij}(\mathbf{x},t) + \gamma(\xi) \int_B \alpha(|\mathbf{x}-\mathbf{x}'|,\xi)\{\lambda\varepsilon_{kk}(\mathbf{x}',t)\delta_{ij} + 2\mu\varepsilon_{ij}(\mathbf{x}',t)\} \, dv(\mathbf{x}'). \tag{12}$$

We assume that the interaction kernel $\alpha(|\mathbf{x}-\mathbf{x}'|,\xi)$ is a continuous and square integrable function in $B\times B$ for each value of ξ *interaction parameter*.

The functions $U_i(\mathbf{x})$, $V_i(\mathbf{x})$ which are defined by

$$U_i(\mathbf{x}) = u_i(\mathbf{x},0),\quad V_i(\mathbf{x}) = \dot{u}_i(\mathbf{x},0)\quad \mathbf{x}\in B \tag{13}$$

are the initial conditions for displacement and velocity fields, respectively.

We define the boundary conditions to be the displacement boundary condition

$$\hat{U}_i(\mathbf{x},t) = u_i(\mathbf{x},t)\quad \mathbf{x}\in S_u,\quad -\infty < t < \infty \tag{14}$$

and the traction boundary condition

$$\hat{T}_i(\mathbf{x},t) = t_i(\mathbf{x},t) = t_{ij}(\mathbf{x},t)n_j(\mathbf{x})\quad \mathbf{x}\in S_t,\quad -\infty < t < \infty \tag{15}$$

where S_u and S_t are the non-overlapping complementary parts of S.

We define the solution of an initial-boundary value problem of non-local elasticity as being to find a triplet $\{u_i(\mathbf{x},t),\varepsilon_{ij}(\mathbf{x},t),t_{ij}(\mathbf{x},t)\}$ which satisfies the field equations (10)–(12), initial conditions (13), and the boundary conditions (14), (15).

4. POSITIVE DEFINITENESS OF THE TOTAL STRAIN ENERGY

In this section the positive definiteness of the total strain energy of homogeneous, isotropic, linear, nonlocal elasticity will be investigated. The positive definiteness of total strain energy has a very important role in many cases. If this property fails it can be easily shown that a body has some equilibrium positions energetically identical which are physically impossible.

The total strain energy at a time t is defined as follows.

$$W(t) = \int_{-\infty}^{t}\int_{B} t_{ij}(\mathbf{x},\tau)\dot{\varepsilon}_{ij}(\mathbf{x},\tau)\,\mathrm{d}v(\mathbf{x})\,\mathrm{d}t,\quad -\infty < t < \infty. \tag{16}$$

Introducing the constitutive equation (12) into (16)

$$\begin{aligned} W(t) = {} & \int_{-\infty}^{t}\int_{B}\{\lambda\varepsilon_{kk}(\mathbf{x},\tau)\dot{\varepsilon}_{ll}(\mathbf{x},\tau)+2\mu\varepsilon_{ij}(\mathbf{x},\tau)\dot{\varepsilon}_{ij}(\mathbf{x},\tau)\}\,\mathrm{d}v\,\mathrm{d}\tau \\ & +\gamma(\xi)\int_{-\infty}^{t}\int_{B}\int_{B}\alpha(|\mathbf{x}-\mathbf{x}'|,\xi)\{\lambda\varepsilon_{kk}(\mathbf{x}',\tau)\dot{\varepsilon}_{ll}(\mathbf{x},\tau) \\ & +2\mu\varepsilon_{ij}(\mathbf{x}',\tau)\dot{\varepsilon}_{ij}(\mathbf{x},\tau)\}\,\mathrm{d}v'\,\mathrm{d}v\,\mathrm{d}\tau \end{aligned} \tag{17}$$

is obtained. If it is assumed that

$$\lim_{t\to-\infty}\varepsilon_{ij}(\mathbf{x},t)\to 0\quad \mathbf{x}\in B \tag{18}$$

then it can be easily shown that from (17)

$$\begin{aligned} 2W(t) = {} & \int_{B}\{\lambda\varepsilon_{kk}(\mathbf{x},t)\varepsilon_{ll}(\mathbf{x},t)+2\mu\varepsilon_{ij}(\mathbf{x},t)\varepsilon_{ij}(\mathbf{x},t)\}\,\mathrm{d}v+\gamma(\xi)\int_{B}\int_{B}\alpha(|\mathbf{x}-\mathbf{x}'|,\xi) \\ & \times\{\lambda\varepsilon_{kk}(\mathbf{x}',t)\varepsilon_{ll}(\mathbf{x},t)+2\mu\varepsilon_{ij}(\mathbf{x}',t)\varepsilon_{ij}(\mathbf{x},t)\}\,\mathrm{d}v'\,\mathrm{d}v \end{aligned} \tag{19}$$

can be obtained. Let us consider the orthonormal base $\{\Psi_i(\mathbf{x})\}_{i=0}^{\infty}$ which is complete in

$L_2(B)$ and also contains the eigenfunctions $\{\Phi_j(\mathbf{x})\}_{j=0}^{\infty}$ of the interaction kernel $\alpha(|\mathbf{x}-\mathbf{x}'|, \xi)$. According to (7) and (8) we can write

$$\varepsilon_{ij}(\mathbf{x}, t) = \sum_{n=0}^{\infty} E_{ij}^{(n)}(t)\Psi_n(\mathbf{x}) \tag{20}$$

where

$$E_{ij}^{(n)}(t) = \int_B \varepsilon_{ij}(\mathbf{x}, t)\Psi_n(\mathbf{x})\, \mathrm{d}v. \tag{21}$$

Substituting (19) into (18) we arrive at

$$2W(t) = \sum_{n=0}^{\infty} \{\lambda E_{mm}^{(n)}(t)E_{kk}^{(n)}(t) + 2\mu E_{ij}^{(n)}(t)E_{ij}^{(n)}(t)\}$$
$$+\gamma(\xi) \sum_{n=0}^{\infty} \alpha_n^{-1}\{\lambda E_{mm}^{(n)}(t)E_{kk}^{(n)}(t) + 2\mu E_{ij}^{(n)}(t)E_{ij}^{(n)}(t)\}. \tag{22}$$

It should be noted that the terms appearing in the first sum are the generalized Fourier coefficients of the functions $\varepsilon_{ij}(\mathbf{x}, t)$ with respect to the complete orthonormal base $\{\Psi_i(\mathbf{x})\}_{i=0}^{\infty}$. But the terms appearing in the second sum are calculated via the set $\{\Phi_i(\mathbf{x})\}_{i=0}^{r}$ which consists of the eigenfunctions of the kernel. Since we have assumed that the set $\{\Phi_i(\mathbf{x})\}_{i=0}^{r}$ is a subset of the complete base $\{\Psi_j(\mathbf{x})\}_{j=0}^{\infty}$ the expression (21) is the sum of two different kinds of terms. One of them is

$$Q_{k1}(t) = \lambda E_{mm}^{(k)}(t)E_{nn}^{(k)}(t) + 2\mu E_{ij}^{(k)}(t)E_{ij}^{(k)}(t) \tag{23}$$

and the other is

$$Q_{k2}(t) = \{1+\gamma(\xi)\alpha_k^{-1}\}\{\lambda E_{mm}^{(k)}(t)E_{nn}^{(k)}(t) + 2\mu E_{ij}^{(k)}(t)E_{ij}^{(k)}(t)\}. \tag{24}$$

It can be easily shown that if

$$3\lambda + 2\mu > 0, \quad \mu > 0 \tag{25}$$

then the quadratic form given by (23) is positive definite. On the other hand, if, in addition to the conditions (25), the condition

$$1+\gamma(\xi)\alpha_k^{-1} > 0 \tag{26}$$

is also satisfied it is clear that the quadratic form given by (23) is also positive definite. In this case, the expression (22) is the sum of positive terms. Consequently, we arrive at the following result. *Let the interaction kernel $\alpha(|\mathbf{x}-\mathbf{x}'|, \xi)$, be a continous function in $L_2(B\times B)$ with the eigenvalues $\alpha_k(k = 1, 2, \ldots)$. If the elastic constants λ, μ and the eigenvalues of the interaction kernel $\alpha_k(k = 1, 2, \ldots)$ obey the following inequalities,*

$$3\lambda+2\mu > 0, \quad \mu > 0, \quad 1+\gamma(\xi)\alpha_k^{-1} > 0, \quad k = 1, 2, \ldots \tag{27}$$

then the total strain energy defined by (16) *is positive definite, i.e.*

$$W(t) > 0 \quad \text{for all } t. \tag{28}$$

5. UNIQUENESS OF THE SOLUTIONS

In this section we wish to show that the positive definiteness of the total strain energy assures the uniqueness of the initial-boundary value problems defined in this study. To this end, let us assume that we have been able to find two different solutions for an initial-boundary value problem, $\{u_i'(\mathbf{x},t), \varepsilon_{ij}'(\mathbf{x},t), t_{ij}'(\mathbf{x},t)\}$, and $\{u_i''(\mathbf{x},t), \varepsilon_{ij}''(\mathbf{x},t), t_{ij}''(\mathbf{x},t)\}$. Because of the linearity the *difference solution* defined by

$$u_i = u_i' - u_i'', \quad \varepsilon_{ij} = \varepsilon_{ij}' - \varepsilon_{ij}'', \quad t_{ij} = t_{ij}' - t_{ij}'' \tag{29}$$

will be the solution corresponding to the null data

$$\begin{aligned} f_i(\mathbf{x},t) &\equiv 0 \quad \mathbf{x} \in B, \quad -\infty < t < \infty \\ U_i(\mathbf{x}) &\equiv 0, \quad V_i(\mathbf{x}) \equiv 0 \quad \mathbf{x} \in B \\ \hat{U}_i(\mathbf{x},t) &\equiv 0 \quad \mathbf{x} \in S_u, \quad \hat{T}_i(\mathbf{x},t) \equiv 0 \quad \mathbf{x} \in S_t, \quad -\infty < t < \infty. \end{aligned} \tag{30}$$

Now, let us multiply the equations of motions (11) by $\dot{u}_i(\mathbf{x},t)$ and integrate over the body

$$\int_B t_{ij,j}(\mathbf{x},t)\dot{u}_i(\mathbf{x},t)\,\mathrm{d}v + \int_B f_i(\mathbf{x},t)\dot{u}_i(\mathbf{x},t)\,\mathrm{d}v = \int_B \rho(\mathbf{x})\ddot{u}_i(\mathbf{x},t)\dot{u}_i(\mathbf{x},t)\,\mathrm{d}v. \tag{31}$$

If we employ the Green Gauss identity and remember that the stress tensor is symmetric then we obtain

$$\int_B t_{ij,j}(\mathbf{x},t)\dot{u}_i(\mathbf{x},t)\,\mathrm{d}v = \int_S t_{ij}(\mathbf{x},t)\dot{u}_i(\mathbf{x},t)n_j(\mathbf{x},t)\,\mathrm{d}a - \int_B t_{ij}(\mathbf{x},t)\dot{\varepsilon}_{ij}(\mathbf{x},t)\,\mathrm{d}v. \tag{32}$$

Substituting (32) into (31)

$$\int_B t_{ij}(\mathbf{x},t)\dot{\varepsilon}_{ij}(\mathbf{x},t)\,\mathrm{d}v + \int_B \rho(\mathbf{x})\ddot{u}_i(\mathbf{x},t)\dot{u}_i(\mathbf{x},t)\,\mathrm{d}v$$
$$= \int_B f_i(\mathbf{x},t)\dot{u}_i(\mathbf{x},t)\,\mathrm{d}v + \int_S t_{ij}(\mathbf{x},t)\dot{u}_i(\mathbf{x},t)n_j(\mathbf{x},t)\,\mathrm{d}a \tag{33}$$

can be written. This equation can be easily integrated with respect to time:

$$\int_B \{\lambda\varepsilon_{kk}(\mathbf{x},t)\varepsilon_{ll}(\mathbf{x},t) + 2\mu\varepsilon_{ij}(\mathbf{x},t)\varepsilon_{ij}(\mathbf{x},t)\}\,\mathrm{d}v + \gamma(\xi)\int_B\int_B \alpha(|\mathbf{x}-\mathbf{x}'|,\xi)$$
$$\times\{\lambda\varepsilon_{kk}(\mathbf{x}',t)\varepsilon_{ll}(\mathbf{x},t) + 2\mu\varepsilon_{ij}(\mathbf{x}',t)\varepsilon_{ij}(\mathbf{x},t)\}\,\mathrm{d}v'\,\mathrm{d}v + \int_B \rho(\mathbf{x})\dot{u}_i(\mathbf{x},t)\dot{u}_i(\mathbf{x},t)\,\mathrm{d}v$$
$$= \int_{-\tau}^{t}\int_B f_i(\mathbf{x},t)\dot{u}_i(\mathbf{x},t)\,\mathrm{d}v + \int_{-\tau}^{t}\int_S t_i(\mathbf{x},t)\dot{u}_i(\mathbf{x},t)\,\mathrm{d}a \tag{34}$$

where we have assumed

$$\lim_{t\to-\infty} \varepsilon_{ij}(\mathbf{x},t)\to 0, \quad \lim_{t\to-\infty} u_i(\mathbf{x},t)\to 0 \quad \mathbf{x}\in B \tag{35}$$

and the definition $t_i(\mathbf{x},t) = t_{ij}(\mathbf{x},t)n_j(\mathbf{x},t)$ is used. The first two terms in the left-hand side of (34) are the total strain energy and the third one is the total kinetic energy and the sum of these terms indicates the total energy stored in the body due to *the external load system*, i.e. the load on the boundary and the body forces which appear on the right-hand side of (34). Note that the load on the boundary $(t_i(\mathbf{x},t))$ comes out *on the surface of the body rather than in a layer*. According to (30) for the *difference solution* we have

$$\int_B \{\lambda\varepsilon_{kk}(\mathbf{x},t)\varepsilon_{ll}(\mathbf{x},t)+2\mu\varepsilon_{ij}(\mathbf{x},t)\varepsilon_{ij}(\mathbf{x},t)\}\,\mathrm{d}v+\gamma(\xi)\int_B\int_B \alpha(|\mathbf{x}-\mathbf{x}'|,\xi)$$
$$\times\{\lambda\varepsilon_{kk}(\mathbf{x}',t)\varepsilon_{ll}(\mathbf{x},t)+2\mu\varepsilon_{ij}(\mathbf{x}',t)\varepsilon_{ij}(\mathbf{x},t)\}\,\mathrm{d}v'\,\mathrm{d}v$$
$$+\int_B \rho(\mathbf{x})\dot{u}_i(\mathbf{x},t)\dot{u}_i(\mathbf{x},t)\,\mathrm{d}v = 0. \tag{36}$$

The positiveness of the kinetic energy is a consequence of its definition. On the other hand, we have shown that the total strain energy is also positive under the conditions (27). So, we conclude from (34)

$$\varepsilon_{ij}(\mathbf{x},t)\equiv 0 \quad \text{and} \quad \dot{u}_i(\mathbf{x},t)\equiv 0 \quad \mathbf{x}\in B, \quad -\infty<t. \tag{37}$$

Considering the initial and boundary conditions we arrive at the following result. *If the elastic constants and the interaction kernel satisfy the conditions* (27) *then the solutions of an initial-boundary value problem defined by* (10)–(15) *differ from each other only by rigid motions.*

6. CONCLUSIONS

In this study we have shown that the initial-boundary value problems in homogeneous, isotropic, linear, nonlocal elasticity defined by (10)–(15) possess only one solution under the condition (27). In a previous work (Altan, 1984) we have given an analogous theorem in which the stress–strain relation was of the form

$$t_{ij}(\mathbf{x},t) = \int_B \alpha(|\mathbf{x}-\mathbf{x}'|)\{\lambda\varepsilon_{kk}(\mathbf{x}',t)\delta_{ij}+2\mu\varepsilon_{ij}(\mathbf{x}',t)\}\,\mathrm{d}v(\mathbf{x}') \tag{38}$$

and we have shown that if the interaction kernel is positive definite (i.e. all of the eigenvalues are positive and the eigenfunctions constitute a complete orthonormal base in the square integrable function space) then the initial-boundary value problems may have only one solution. As is clear from this study if the constitutive relation of nonlocal elasticity is replaced by

$$t_{ij}(\mathbf{x},t) = \lambda\varepsilon_{kk}(\mathbf{x},t)\delta_{ij}+2\mu\varepsilon_{ij}(\mathbf{x},t)+\int_B \beta(|\mathbf{x}-\mathbf{x}'|)\{\lambda\varepsilon_{kk}(\mathbf{x}',t)\delta_{ij}+2\mu\varepsilon_{ij}(\mathbf{x}',t)\}\,\mathrm{d}v(\mathbf{x}') \tag{39}$$

then the internal kernel does not necessarily need to be a positive definite kernel to ensure the uniqueness of the solutions of the initial-boundary value problems considered in this study. Of course, this is an important difference, because the interaction kernel $\beta(|\mathbf{x}-\mathbf{x}'|,\xi)$ for the constitutive relation in the form (39) can be chosen from a wider class of functions. But we believe that the essential difference between the constitutive relations (38) and (39) can be clarified by constructing an existence theory for the boundary value problems in nonlocal elasticity.

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